Solutions to Test 3

Q1. (a). Suppose $\mathcal{C} = \{U_\alpha \mid \alpha \in I\}$ is an open cover of A.

Choose an non-empty U_{α_0}

Then $A \setminus U_{\alpha_0} = \{a_1, a_2, \dots, a_N\}$

For each $a_i \in A \setminus U_{\alpha_0}$

choose $\alpha_i \in I$ s.t. $a_i \in U_{\alpha_i}$

Then $\{U_{\alpha_0}, U_{\alpha_1}, \dots, U_{\alpha_N}\}$

is a finite subcover of $\mathcal{C}$

(b). Case 1: $\# X = +\infty$

X is connected.

Otherwise, $\exists \ S \subset X$

s.t. S and S^c are non-empty open set

$\Rightarrow \# S^c < +\infty$

$\# S < +\infty$

$\Rightarrow \# X < +\infty$ Contradiction !

<u>Case 2</u> $\quad 2 \leq \# X < +\infty$

X is NOT connected

Because X is discrete

$$X = \{a\} \cup \{b, \dots\}$$

open

<u>Case 3</u>: $\quad \# X = 1$

X is connected.

Q2. (a). NO.

$$\mathbb{R} = (-\infty, 0) \cup [0, +\infty)$$

Open in $\mathbb{R}_\ell$

(b). All connected subsets are sets consist of 1 pt or $\emptyset$

Pf: If $a, b \in A$ with $a < b$

Then $\exists\, c$

s.t. $a < c < b$ $(\cdots)$

$\Rightarrow \emptyset \neq (-\infty, c) \cap A$ open in A

$\emptyset \neq [c, +\infty) \cap A$ open in A

$\Rightarrow$ A is NOT connected.

(c). Consider the open cover

$$\mathcal{C} = \left\{ [0, 1 - \tfrac{1}{n}) \;\middle|\; n = 2, 3 \cdots \right\} \cup \{ [1, 2) \}$$

of $[0, 1]$

It has NO finite sub cover.

(d). NO.

Consider $A \triangleq \{1-\frac{1}{n} \mid n=1,2,3,\cdots\} \subset [0,1]$

<u>Claim</u>: A has NO accumulation pt.

<u>Pf</u>: <u>case 1</u>: $1-\frac{1}{n} \leq a < 1-\frac{1}{n+1}$ for an $n \in \mathbb{Z}^+$

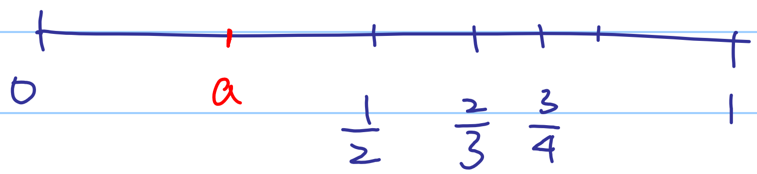
$$\left[a,\ a+\frac{1-\frac{1}{n+1}-a}{2}\right) \setminus \{a\} \cap A = \emptyset$$

$\Rightarrow$ a is NOT an accumulation pt.

<u>case 2</u> $a = 1$

$$[a,\ a+1) \setminus \{a\} \cap A = \emptyset$$

$\Rightarrow$ a is NOT an accumulation pt.



Q3. Let $\pi_D : \mathbb{R}_D \times \mathbb{R} \longrightarrow \mathbb{R}_D$

$\pi : \mathbb{R}_D \times \mathbb{R} \longrightarrow \mathbb{R}$

be projections.

(a). $L_{p,q}$ is cpt $\iff \pi_D(p) = \pi_D(q)$

Pf : • If $\pi_D(p) = \pi_D(q)$

Then $\pi|_{L_{p,q}} : L_{p,q} \longrightarrow \operatorname{Im}(\pi|_{L_{p,q}})$

$\qquad\qquad\qquad\qquad\qquad\qquad ||$

$\qquad\qquad\qquad\qquad\qquad$ closed interval in $\mathbb{R}_{std}$

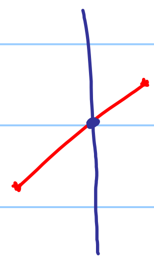
is homemorphism.

closed interval in $\mathbb{R}_{std}$ is cpt $\Longrightarrow L_{p,q}$ is cpt

• If $\pi_D(p) \neq \pi_D(q)$

Then for $\forall$ $t_0 \in [0,1]$

we have



$(1-t_0)\vec{p} + t_0\vec{q}$

$= L_{p,q} \cap \underbrace{\pi_D\left((1-t_0)\vec{p} + t_0\vec{q}\right) \times \mathbb{R}}_{\text{open in } \mathbb{R}_D \times \mathbb{R}}$

$\Rightarrow$ $\forall$ pt in $L_{p.q}$ is open

$\Rightarrow$ $L_{p.q}$ is discrete w/ $\# L_{p.q} = +\infty$

$\Rightarrow$ $L_{p.q}$ is NOT cpt

(consider the open cover
$\mathcal{C} = \{ \{a\} \mid a \in L_{p.q} \}$)

b). If $L_{p.q}$ is NOT cpt

then $L_{p.q}$ is discrete w/ $\# L_{p.q} = +\infty$

Consider the seq. $x_n = (1-\frac{1}{n})\vec{p} + \frac{1}{n}\vec{q}$
which satisfy $x_m \neq x_n$ for $\forall$ $m \neq n$

If $x_{n_k} \longrightarrow x$ in $L_{p.q}$

then $\exists.\ K \in \mathbb{Z}^+$
s.t. $k \geq K \Rightarrow x_{n_k} = x$

contradiction!

(c). $\emptyset \neq A$ is connected $\iff$ $\pi_1(A)$ = constant
$\pi_2(A)$ is an interval

Pf: [$\Leftarrow$] If $\pi_D(A)$ = constant
$\pi(A)$ is an interval

then $\pi|_A : A \longrightarrow$ interval

is homemorphism

So A is connected.

[$\Rightarrow$] If $\emptyset = A$ is connected

Assume $\exists\ (a_1, b_1), (a_2, b_2) \in A$

s.t. $a_1 < a_2$

Then $(-\infty, a_1] \times \mathbb{R}^1$ & $(a_1, +\infty) \times \mathbb{R}^1$

disconnects A.

$\Longrightarrow$ A is NOT connected

So $\pi_D(A)$ = one pt $\triangleq a$

$\Longrightarrow$ $A = a \times D$ where $D \subset \mathbb{R}_{std}$

A is conn. $\Longrightarrow$ D is conn. in $\mathbb{R}$ std

$\Longrightarrow$ D is interval.

Q4. (a). $X = (0, 1)$

$Y = \mathbb{R}^1$

$f(x) = x$

(b). • f is cts, C is conn,

$\Longrightarrow$ $f(C)$ is conn.

• Assume $\exists$ conn. $D \subset Y$

s.t. $f(C) \subsetneq D$

f^{-1} is cts D is conn.

$\Longrightarrow$ $f^{-1}(D)$ is conn.

But $f(C) \subsetneq D$ & f is bij.

So $f^{-1}(f(C)) \subsetneq f^{-1}(D)$

$$\overset{\displaystyle f^{-1}(f(C))}{\underset{\displaystyle C}{\shortparallel}}$$

So C is NOT conn. component.

Contradiction!

(c). $f : X \to Y$ is homemorphism

$\Rightarrow$ $f|_{X \setminus S} : X \setminus S \longrightarrow Y \setminus f(S)$

is homemorphism.

where $X \setminus S$ & $Y \setminus f(S)$ have subspace topo.

If $X \setminus S = U \sqcup V$ is disconnected.

then $Y \setminus f(S) = f(U) \sqcup f(V)$ is also disconnected.

5. (a) If $\bigcap_{n=1}^{\infty} F_n = \emptyset$

Then $\{ F_n^c \}$ is open cover of X

X is cpt $\Rightarrow$

we have a finite sub cover

$$\{ F_{n_1}^c , \dots F_{n_k}^c \}$$

$$\Rightarrow \quad X = \bigcup_{i=1}^{k} F_{n_i}^c$$

$$\Rightarrow \quad \phi = \bigcap_{i=1}^{k} F_{n_i} = F_{\max\{n_1, \cdots, n_k\}}$$

Contradiction !

(b). X is Hausdorff & A_k is cpt

$\Rightarrow$ A_k is closed for $k=1, \cdots, n$

$\Rightarrow$ $\bigcap_{k=1}^{n} A_k$ is closed in X

$\Rightarrow$ $\bigcap_{k=1}^{n} A_k$ is closed in cpt A_1

$\Rightarrow$ $\bigcap_{k=1}^{n} A_k$ is cpt.

(c). $A = [-1, 1]$

$B = [-1, 0) \cup \{\infty\} \cup (0, 1]$

It is easy to show :

A & B homeomorphic to the closed interval $[-1, 1]$ in $\mathbb{R}_{std}$.

So A & B are cpt

But $A \cap B = [-1, 0) \cup (0, 1]$ in $\mathbb{R}_e$

is homemorphic to $[-1, 0) \cup (0, 1]$

in $\mathbb{R}_{std}$.

So $A \cap B$ is not cpt.